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Recent Advances in the Use of Separation of Variables Methods in General Relativity

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Recent advances in the use of separation of variables methods in general relativity

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This review article is a guide to work that uses the method of separation of variables for problems that occur in general relativity. The main emphasis is on recent progress in the solution of important systems of equations such as Dirac's equation, Maxwell's equations and the gravitational perturbation (or spin 2) equations. Recent advances and established results for these equations in Kerr black hole and Robertson–Walker space-time backgrounds form the central theme of the discussion. These two important physical examples also illustrate some of the difficulties in a theory of solution by separation of variables methods for systems of equations. Other aspects of this subject such as solutions of the Rarita–Schwinger equation (spin $\frac{3}{2}$) and the role of generalized Hertz potentials are also discussed.

1. Introduction and scalar equations

The method of separation of variables is a well-known procedure in classical and quantum mechanics. It enables the solution of many of the fundamental (scalar) partial differential equations of mathematical physics to be reduced to the solution of ordinary differential equations. Typically one tries to represent a solution of the original problem as a sum or product of functions, each term depending on a single coordinate. This is frequently successful for the Hamilton–Jacobi, Helmholtz and Schrödinger equations. A comprehensive theory exists for which a review has been given by Miller (1988). The aims of the theory are twofold: first to obtain explicit solutions of partial differential equations and second, to find intrinsic characterizations of the separable coordinates, separation parameters and frames that occur in the solutions. The natural place for these methods as applied to scalar problems in general relativity is in the solution of the equations for the geodesics. An obvious question to ask is for what space times is it possible to solve the Hamilton–Jacobi equation for the geodesics, $g^{ij}\partial_{x^i}S\partial_{x^j}S = \lambda$, by separation of variables? Although this problem has not been comprehensively solved, a significant contribution was made by Carter (1968). He investigated spaces with a two-parameter abelian isometry group for which the Hamilton–Jacobi equation for the geodesics is soluble by additive separation of variables in such a way that a certain natural canonical orthonormal frame is determined. Carter included in his studies the extension of the classical separation of variables ideas to include charged particles in the presence of an electromagnetic field. Through imposition of only the existence of a two-dimensional abelian isometry group a detailed study was possible. The additional requirement that the Schrödinger equation also separates via the product

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separation ansatz further made the complete characterization for such systems possible. These requirements lead to a metric of the form

$$ds^2 = \frac{Z}{\Delta_\lambda}d\lambda^2 + \frac{Z}{\Delta_\mu}d\mu^2 + \frac{\Delta_\mu}{Z}[P_\lambda\,d\psi - Q_\lambda\,d\sigma]^2 + \frac{\Delta_\lambda}{Z}[P_\mu\,d\psi - Q_\mu\,d\sigma]^2,$$

where ψ and σ are ignorable variables (i.e. the components of the metric tensor do not depend on these coordinates) and λ and μ are the two non-ignorable variables. Here Δ_κ, P_κ, Q_κ are functions of κ for $\kappa = \lambda$, μ. The function Z is defined by $Z = P_\lambda Q_\mu - P_\mu Q_\lambda$ where $\partial^2 Z/\partial\lambda\,\partial\mu = 0$.

The simplest electromagnetic field which is compatible with separability of the Hamilton–Jacobi equation in these coordinates is a covariant vector potential of the form

$$A = Z^{-1}(P_\lambda X_\mu + P_\mu X_\lambda)\,d\psi - Z^{-1}(Q_\lambda X_\mu + Q_\mu X_\lambda)\,d\sigma$$

with X_κ functions of κ as above. The complete solution of the Einstein vacuum equations (with and without a cosmological term) and the Einstein–Maxwell equations for these restrictions results in four classes of solutions.

[A] $$ds^2 = (\lambda^2+\mu^2)[d\lambda^2/\Delta_\lambda + d\mu^2/\Delta_\mu] + (\Delta_\mu[d\chi - \lambda^2\,d\psi]^2 - \Delta_\lambda[d\chi + \mu^2\,d\psi]^2)/(\lambda^2+\mu^2),$$

where

$$\Delta_\lambda = \tfrac{1}{3}\Lambda\lambda^4 + h\lambda^2 - 2m\lambda + p + e^2, \quad \Delta_\mu = \tfrac{1}{3}\Lambda\mu^4 - h\mu^2 + 2q\mu + p,$$

$$A = e\left[\frac{\lambda\mu(\nu\cos\alpha + \lambda\sin\alpha)}{\lambda^2+\mu^2}d\psi + \frac{(\lambda\cos\alpha - \mu\sin\alpha)}{\lambda^2+\mu^2}d\chi\right],$$

[$\bar{\mathrm{B}}(+)$] $$ds^2 = (\lambda^2+l^2)[d\lambda^2/\Delta_\lambda + d\mu^2/\Delta_\mu + \Delta_\mu\,d\psi^2] - \Delta_\lambda[d\chi + 2l\mu\,d\psi]^2/(\lambda^2+l^2),$$

where

$$\Delta_\lambda = \Lambda(\tfrac{1}{3}\lambda^4 + 2l^2\lambda^2 - l^4) + h(\lambda^2 - l^2) - 2m\lambda + e^2, \quad \Delta_\mu = -h\mu^2 + 2q\mu + p,$$

$$A = e\left[\frac{\mu[2l\cos\alpha + (\lambda^2+l^2)\sin\alpha]}{\lambda^2+l^2}d\psi + \frac{\lambda\cos\alpha}{\lambda^2+l^2}d\chi\right],$$

[$\bar{\mathrm{B}}(-)$] $$ds^2 = (\mu^2+k^2)[d\lambda^2/\Delta_\lambda + d\mu^2/\Delta_\mu - \Delta_\lambda\,d\psi^2] + \Delta_\mu[d\chi - 2k\lambda\,d\psi]^2/(\mu^2+k^2),$$

where

$$\Delta_\lambda = h\lambda^2 - 2m\lambda + n, \quad \Delta_\mu = \Lambda(\tfrac{1}{3}\mu^4 + 2k^2\mu^2 - k^4) - h(\mu^2 - k^2) + 2q\mu - e^2,$$

$$A = e\left[\frac{\lambda[(\mu^2+k^2)\cos\alpha + 2k\mu\sin\alpha]}{\mu^2+k^2}d\psi + \frac{\mu\sin\alpha}{\mu^2+k^2}d\chi\right],$$

[D] $$ds^2 = d\lambda^2/\Delta_\lambda + d\mu^2/\Delta_\mu + \Delta_\mu\,d\chi^2 - \Delta_\lambda\,d\psi^2,$$

where

$$\Delta_\lambda = (\Lambda + e^2)\lambda^2 - 2m\lambda + n, \quad \Delta_\mu = (\Lambda + e^2)\mu^2 - 2q\mu + p,$$

$$A = e\{\lambda\cos\alpha\,d\psi - \mu\sin\alpha\,d\chi].$$

Other results are available on this topic, as for instance in Boyer *et al.* (1981).

2. Separation of variables in non-scalar equations

The theory for the solution of systems of equations by methods that directly or indirectly relate to separation of variables has remained unclear in general relativity. However, from a knowledge of group theory it is straightforward to obtain a decoupling, in spherical coordinates, of the equations associated with the

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Schwarzschild metric, because of the inherent spherical symmetry of this metric (Gel'fand *et al.* 1963). This is also true for Robertson–Walker cosmological models which admit six-dimensional isometry groups (Kalnins & Miller 1991). The first results that indicated that separation of variables is possible for a Kerr space-time were those due to Teukolsky (1973). Teukolsky showed that certain components of the gravitational perturbations of a Kerr black hole were solvable by the means of the separation of variables method. (The gravitational perturbation equations (11) describe spin 2 particles with zero mass (Lifshitz & Khalatnikov 1963).) Furthermore his analysis extended to the spinor Maxwell and neutrino field equations. These ideas have been further extended to the solution of the Dirac equation in an Einstein–Maxwell space time by Kamran & McLenaghan (1984*a*, *b*). One of the crucial steps was the solution of the Dirac equation in the space time of a Kerr black hole, achieved by Chandrasekhar (1976). This result was later extended by Page (1976) to the case of an electron in the Kerr–Newman space-time background of a charged black hole. In this article we also show that the separation also works in the presence of a magnetic charge g. The expression for the infinitesimal distance in this case is (in spherical coordinates)

$$\mathrm{d}s^2 = (1-2Mr/\rho^2)\,\mathrm{d}t^2 - \rho^2(\mathrm{d}r^2/\Delta + \mathrm{d}\theta^2) \\ -((r^2+a^2)+(2a^2Mr/\rho^2)\sin^2\theta)\sin^2\theta\,\mathrm{d}\varphi^2 + (4aMr/\rho^2)\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi,$$

where $\Delta = r^2+a^2+q^2+g^2-2Mr$, $\bar{\rho} = r+\mathrm{i}a\cos\theta$, $\rho^2 = \overline{\rho\rho^*}$ and the associated electromagnetic field has the components

$$A = \{qr[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi] + g\cos\theta[a\,\mathrm{d}t-(r^2+a^2)\,\mathrm{d}\varphi]\}/\rho^2$$

This metric corresponds to the Kerr–Newman black hold solution when $g=0$ and the Kerr solution when $q=0$ also. Associated with this metric is the null tetrad of vectors given by

$$l_i\,\mathrm{d}x^i = \Delta^{-1}(\Delta\,\mathrm{d}t - \rho^2\,\mathrm{d}r - a\Delta\sin^2\theta\,\mathrm{d}\phi), \quad n_i\,\mathrm{d}x^i = (1/2\rho^2)(\Delta\,\mathrm{d}t+\rho^2\,\mathrm{d}r - a\Delta\sin^2\theta\,\mathrm{d}\phi),$$
$$m_i\,\mathrm{d}x^i = (1/\bar{\rho}\sqrt{2})(\mathrm{i}a\sin\theta\,\mathrm{d}t - \rho^2\,\mathrm{d}\theta - \mathrm{i}(r^2+a^2)\sin\theta\,\mathrm{d}\phi).$$

The null tetrad is used when physical equations are rewritten in spinor form using the null tetrad formalism as expounded in Penrose & Rindler (1984). (These vectors are aligned with the principal null directions of the Weyl spinor.) The non-zero spin coefficients are

$$\rho = -\frac{1}{\bar{\rho}^*}, \quad \beta = \frac{\cot\theta}{2\sqrt{2\bar{\rho}}}, \quad \pi = \frac{\mathrm{i}a\sin\theta}{\sqrt{2(\bar{\rho}^*)^2}}, \quad \tau = -\frac{\mathrm{i}a\sin\theta}{\sqrt{2\rho^2}},$$
$$\mu = -\Delta/2\rho^2\bar{\rho}^*, \quad \gamma = \mu + (r-M)/2\rho^2, \quad \alpha = \pi - \beta^*.$$

The only non-zero component of the Weyl spinor is $\Psi_2 = -M/(\bar{\rho}^*)^3$.

The key observation of Teukolsky was obtained by examining the linear perturbation equations $\delta R_{\alpha\beta} = 0$ for a Kerr black hole. From the Bianchi and Ricci identities the following relations hold true

$$(\delta^* - 4\alpha + \pi)\,\hat{\Psi}_0 - (D-4\rho^*)\,\hat{\Psi}_1 = 3\hat{\kappa}\Psi_2, \quad (\Delta - 4\gamma+\mu)\,\hat{\Psi}_0 - (\delta - 4\tau - 2\beta)\,\hat{\Psi}_1 = 3\hat{\sigma}\Psi_2$$
$$(D-\rho-\rho^*)\,\hat{\sigma} - (\delta - \tau + \pi^* - \alpha^* - 3\beta)\,\hat{\kappa} = \hat{\Psi}_0,$$
$$(\delta + 4\beta - \rho^*)\,\hat{\Psi}_4 - (\Delta + 2\gamma + 4\mu)\,\hat{\Psi}_3 = -3\hat{\nu}\Psi_2,$$
$$(D-\rho^*)\,\hat{\Psi}_4 - (\delta^* + 4\pi + 2\alpha)\,\hat{\Psi}_3 = -3\hat{\lambda}\Psi_2,$$
$$(\Delta + \mu + \mu^* + 3\gamma - \gamma^*)\,\hat{\lambda} - (\delta^* + 3\alpha + \beta^* + \pi - \tau^*)\,\hat{\nu} = -\hat{\Psi}_4,$$

where $D = l^i \partial_{x^i}$, $\Delta = n^i \partial_{x^i}$, and $\delta = m^i \partial_{x^i}$. (In these equations $\hat{\Psi}_0$, $\hat{\Psi}_1$, $\hat{\Psi}_3$, $\hat{\Psi}_4$, $\hat{\sigma}$, $\hat{\kappa}$, $\hat{\nu}$ and $\hat{\lambda}$ are first-order components of the Weyl spinor and the spin coefficients, respectively.) As a result of the identities that result from the commutation relations of the directional derivatives the perturbed components of the Weyl spinor Ψ_0 and Ψ_4 satisfy the decoupled equations

$$[(D-2\rho-\rho^*)(\Delta-4\gamma+\mu)-(\delta-2\tau+\pi^*-\alpha^*-3\beta)(\delta^*-4\alpha+\pi)]\,\hat{\Psi}_0 = \Psi_2\,\hat{\Psi}_0,$$
$$[(\Delta+2\mu+\mu^*+3\gamma-\gamma^*)(D-\rho)-(\delta^*+3\alpha+\beta^*+2\pi-\tau^*)(\delta+4\beta-\tau)]\,\hat{\Psi}_4 = \Psi_2\,\hat{\Psi}_4.$$

When $\Phi_4\,\mathrm{e}^{\mathrm{i}(\sigma t+m\varphi)} = (\bar{\rho})^4\,\Psi_4$, $\Phi_0\,\mathrm{e}^{\mathrm{i}(\sigma t+m\varphi)} = \Psi_0$, Teukolsky deduced that these equations can be reduced to the forms

$$[\Delta D_1 D_2^\dagger + L_{-1}^\dagger L_2 - 6\mathrm{i}\sigma(r+\mathrm{i}a\cos\theta)]\,\Phi_0 = 0,$$
$$[\Delta D_{-1}^\dagger D_0 + L_{-1} L_2^\dagger + 6\mathrm{i}\sigma(r+\mathrm{i}a\cos\theta)]\,\phi_4 = 0,$$

where

$$L_n = \partial/\partial\theta + Q + n\cot\theta,\quad L_n^\dagger = \partial/\partial\theta - Q + n\cot\theta,\quad Q = \sigma a\sin\theta + m\,\mathrm{cosec}\,\theta,$$

$$D_n = \frac{\partial}{\partial r} + \mathrm{i}\frac{K}{\Delta} + \frac{2n(r-M)}{\Delta},\quad D_n^\dagger = \frac{\partial}{\partial r} - \mathrm{i}\frac{K}{\Delta} + \frac{2n(r-M)}{\Delta},\quad K = (r^2+a^2)\,\sigma + am.$$

From these equations we can see that a solution of the separated form $\Phi_0 = R_2(r)\,S_2(\theta)$ and $\Phi_4 = R_{-2}(r)\,S_{-2}(\theta)$ is possible if the functions $R_{\pm 2}(r)$, $S_{\pm 2}(\theta)$ satisfy the ordinary differential equations

$$(\Delta D_1 D_2^\dagger - 6\mathrm{i}\sigma r)\,R_2 = \lambda R_{-2},\quad (L_{-1}^\dagger L_2 + 6a\sigma\cos\theta)\,S_2 = -\lambda S_2,$$
$$(\Delta D_{-1}^\dagger D_0 + 6\mathrm{i}\sigma r)\,R_{-2} = \lambda R_{-2},\quad (L_{-1} L_2^\dagger - 6a\sigma\cos\theta)\,S_{-2} = -\lambda S_{-2}.$$

These are the equations for Teukolsky functions of spin 2. Similar results have also been deduced for spin 1 Maxwell's equations and the spin $\frac{1}{2}$ neutrino field. In the appendix Teukolsky functions for general spin are discussed together with some of their properties.

3. Separation of variables and the Dirac equation in curved space-time

Dirac's equation for spin $\frac{1}{2}$ particles with charge e in an electromagnetic field has the form $\gamma^\mu(\nabla_\mu - \mathrm{i}eA_\mu)\,\psi = \mathrm{i}m_e\,\psi$ where $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, and $\{,\}$ is the anticommutator. In spinor form the Dirac equation is

$$(\nabla^B{}_{B'} - \mathrm{i}eA^B{}_{B'})\,\varphi_B = (\mathrm{i}m_\mathrm{e}/\sqrt{2})\,\chi_{B'},\quad (\nabla_B{}^{B'} - \mathrm{i}eA_B{}^{B'})\,\chi_{B'} = -(\mathrm{i}m_\mathrm{e}/\sqrt{2})\,\varphi_B. \tag{1}$$

Restricted to a black hole space time with both electric and magnetic charge and expressed in terms of the modified field components

$$\varphi_0 = \phi_0\,\mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)},\quad \bar{\rho}^*\varphi_1 = \phi_1\,\mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)},\quad \chi_{0'} = X_0\,\mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)},\quad \bar{\rho}\chi_{1'} = X_1\,\mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)},$$

these equations assume the form

$$-(L_{\frac{1}{2}} - (\mathrm{i}eg/\sqrt{2})\cot\theta)\,\phi_0 + (D_0 - \mathrm{i}eqr/\sqrt{2}\Delta)\,\phi_1 = -\mathrm{i}m_\mathrm{e}(r-\mathrm{i}a\cos\theta)\,X_0,$$
$$(\Delta D_{\frac{1}{2}}^\dagger + \mathrm{i}eqr/\sqrt{2})\,\phi_0 + (L_{\frac{1}{2}}^\dagger + (\mathrm{i}eg/\sqrt{2})\cot\theta)\,\phi_1 = -\mathrm{i}m_\mathrm{e}(r-\mathrm{i}a\cos\theta)\,X_1,$$
$$-(L_{\frac{1}{2}}^\dagger + (\mathrm{i}eg/\sqrt{2})\cot\theta)\,X_0 + (D_0 - \mathrm{i}eqr/\sqrt{2}\Delta)\,X_1 = \mathrm{i}m_\mathrm{e}(r+\mathrm{i}a\cos\theta)\,\phi_0,$$
$$(\Delta D_{\frac{1}{2}}^\dagger + \mathrm{i}eqr/\sqrt{2})\,X_0 + (L_{\frac{1}{2}} - (\mathrm{i}eg/\sqrt{2})\cot\theta)\,X_1 = \mathrm{i}m_\mathrm{e}(r+\mathrm{i}a\cos\theta)\,\phi_1.$$

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They can be solved by the separation ansatz

$$\phi_0 = R_{\frac{1}{2}} S_{\frac{1}{2}}, \quad \phi_1 = R_{-\frac{1}{2}} S_{\frac{1}{2}}, \quad X_0 = -R_{\frac{1}{2}} S_{-\frac{1}{2}}, \quad X_1 = R_{-\frac{1}{2}} S_{-\frac{1}{2}}$$

to give the coupled equations

$$(L_{\frac{1}{2}} - (\mathrm{i}eg/\surd 2)\cot\theta)\, S_{\frac{1}{2}} = (\lambda - am_{\mathrm{e}}\cos\theta)\, S_{-\frac{1}{2}}, \quad (D_0 - \mathrm{i}eqr/\surd 2\Delta)\, R_{-\frac{1}{2}} = (\lambda + \mathrm{i}m_{\mathrm{e}}\, r)\, R_{\frac{1}{2}},$$
$$(L^{\dagger}_{\frac{1}{2}} + (\mathrm{i}eg/\surd 2)\cot\theta)\, S_{-\frac{1}{2}} = -(\lambda + am_{\mathrm{e}}\cos\theta)\, S_{\frac{1}{2}}, \quad (\Delta D^{\dagger}_{\frac{1}{2}} + \mathrm{i}eqr/\surd 2)\, R_{\frac{1}{2}} = (\lambda - \mathrm{i}m_{\mathrm{e}}\, r)\, R_{-\frac{1}{2}},$$

where λ is a constant. The basic question left open by this work was how is the separation parameter λ invariantly characterized. This is answered by the existence of a non-trivial symmetry operator associated with the Killing–Yano tensor $K_{AA'BB'}$. In tensorial notation, a Killing–Yano tensor of order p is any skew symmetric tensor $K_{\mu_1\mu_2\ldots\mu_p}$ which satisfies $\nabla_{(\mu}K_{\nu)\mu_2\ldots\mu_p} = 0$. For $p = 2$ this condition, in spinor notation, is equivalent to the conditions

$$\nabla_{(AA'}K_{BC)} = 0, \quad \nabla_{A(A'}\bar{K}_{B'C')} = 0, \quad \nabla_{BA'}K^{B}_{A} + \nabla_{AB'}\bar{K}^{B'}_{A'} = 0,$$

where $$K_{\alpha\beta} = K_{AA'BB'} = \tfrac{1}{2}[\epsilon_{A'B'}K_{AB} + \epsilon_{AB}\bar{K}_{A'B'}]$$

is the decomposition of $K_{\alpha\beta}$ in terms of symmetric Killing spinors K_{AB}, $\bar{K}_{A'B'}$. For Kerr space-time there is only one Killing–Yano tensor of valence two and it has non-zero components $K_{AB} = -\bar{\rho}^{*}$, and $\bar{K}_{A'B'} = \bar{\rho}$.

The corresponding characterization of the separation parameter is given by:

$$\begin{bmatrix} 0 & L_{A}{}^{A'} \\ N^{A}{}_{A'} & 0 \end{bmatrix} \begin{pmatrix} \phi_A \\ -\chi_{A'} \end{pmatrix} = \frac{\lambda}{\surd 2} \begin{pmatrix} \phi_A \\ -\chi_{A'} \end{pmatrix}, \tag{2}$$

where

$$L_{AA'} = K_{AA'}{}^{BB'}\nabla_{BB'} + \tfrac{1}{3}M_{AA'}, \quad N_{AA'} = K_{AA'}{}^{BB'}\nabla_{BB'} - \tfrac{1}{3}M_{AA'}.$$

Here $M_{AA'}$ is a Killing vector obtained from the Killing–Yano tensor via

$$M_{AB'} = \tfrac{1}{3}\nabla^{BA'}K_{AA'BB'}, \quad \nabla_{(A(A'}M_{B)B')} = 0.$$

This result is due to Carter & McLenaghan (1979) though originally expressed in tensor rather than spinor form. Writing the Dirac equation (1) in the form

$$H\begin{pmatrix} \phi_A \\ -\chi_{A'} \end{pmatrix} = \frac{\mathrm{i}m_{\mathrm{e}}}{\surd 2}\begin{pmatrix} \phi_A \\ -\chi_{A'} \end{pmatrix}$$

one can show that the operator on the left-hand side of (2) is a *symmetry operator* for the Dirac equation, i.e. it commutes with H. This operator together with the trivial symmetry operators ∂_ϕ, ∂_t is responsible for the separation. The separation constants are just the eigenvalues of these operators.

In fact the most general first-order formally self-adjoint operator that commutes with the Dirac operator has, in tensor notation, the form (McLenaghan & Spindel 1979; Kamran *et al.* 1988),

$$L = (\mathrm{i}B^{i}I_4 + \mathrm{i}C\gamma^{i} + D^{i}_{j}\gamma^{5}\gamma^{j} + E^{i}_{jk}\gamma^{jk})\nabla_i + RI_4 - \tfrac{3}{4}(\nabla^{*}_{j}E^{j})\gamma^{5} + \tfrac{1}{3}(\nabla^{*}_{k}D^{k}_{j})\gamma^{j} - \tfrac{1}{4}\nabla_k B_j\gamma^{jk}, \tag{3}$$

where C and R are real-valued constants, B_i, D_{ij} and E_{ijk} are Killing–Yano tensors,

$$^{*}D_{ij} = \tfrac{1}{2}\epsilon_{ijkl}D^{kl}, \quad ^{*}E_i = \tfrac{1}{6}\epsilon_{ijkl}E^{jkl}, \quad \gamma^{5} = \tfrac{1}{24}\epsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} \quad \text{and} \quad \gamma^{jk} = \gamma^{j}\gamma^{k} - \gamma^{k}\gamma^{j}.$$

A lesson to be learned from this invariant characterization is that the existence of a Killing–Yano tensor is crucial to the uncoupling of the Dirac equation in a Kerr space-time. An immediate observation is that the notion of separation of variables component by component is not able to describe this method of solution. The occurrence of the factors $\bar{\rho}$ and $\bar{\rho}^*$ in the form of the solution indicate an alternative form of separation of variables. Separation that occurs in this manner in the Dirac equation can be described using the idea of factorizable systems as formulated by Shapovalov & Ékle (1974*a*, *b*), see also Miller (1988).

Definition. A *factorizable system* for the Dirac equation

$$H\Psi \equiv \gamma^{\mu}\nabla_{\mu}\Psi \equiv (H_i\,\partial/\partial x^i + V)\,\Psi = \mathrm{i}m_{\mathrm{e}}\,\Psi \tag{4}$$

is a collection of four first-order partial differential equations of the form

$$\partial\Psi/\partial x^i = (S_{ij}\,\lambda^j - V_i)\,\Psi, \tag{5}$$

where the λ^i are complex parameters, the functions S_{ij}, V_i are 4×4 matrices where the determinant of the 16×16 matrix $[S_{ij}]$ is non-zero and where substitution of (5) into (4) yields an identity. A factorizable system is said to be *separable* in the local coordinates x^l if

$$\partial S_{ij}/\partial x^k = \partial V_i/\partial x^k = 0, \quad \text{for} \quad k \neq i.$$

The significance of this definition is the relation between factorizable systems and complete sets of commuting first-order operators. In fact the integrability conditions

$$[\partial^2/\partial x^i\,\partial x^j - \partial^2/\partial x^j\,\partial x^i]\,\Psi = 0, \quad i \neq j$$

for the existence of a factorizable system are satisfied if and only if there exist functions A^{ik}, B^i such that the first-order partial differential operators

$$A^{(i)} \coloneqq A^{ik}\,\partial/\partial x^k + B^i, \quad i = 1, \ldots, n$$

commute:

$$[A^{(i)}, A^{(j)}] = 0, \quad \text{for} \quad i \neq j,$$

$$H = A^{(1)}, \quad A^{ik}S_{kj} = S_{jk}A^{ki} = \delta^i_j I \quad \text{and} \quad B^i = A^{ik}V_k.$$

Here, I is the 4×4 identity matrix. As a corollary of this definition of a factorizable system, the (separable) solutions Ψ of (5) are simultaneous eigenfunctions of the operators $A^{(i)}$. A question arising from this mechanism is what space-times and coordinate systems permit separation via the factorization method. Debever & McLenaghan (1981) have obtained a very general null tetrad fitting this method and given by

$$l_i\,\mathrm{d}x^i = \tfrac{1}{\sqrt{2}}|Z(z,w)|^{\frac{1}{2}}(g_0^{-2}W(w)^{-1}\,\mathrm{d}w + f_0\,W(w)\,Z(x,w)^{-1}(\epsilon_1\,\mathrm{d}u + m(x)\,\mathrm{d}v),$$

$$n_i\,\mathrm{d}x^i = \tfrac{1}{\sqrt{2}}|Z(z,w)|^{\frac{1}{2}}(-f_0\,g_0^{-2}W(w)^{-1}\,\mathrm{d}w + W(w)\,Z(w,z)^{-1}(\epsilon_1\,\mathrm{d}u + m(x)\,\mathrm{d}v),$$

$$m_i\,\mathrm{d}x^i = \tfrac{1}{\sqrt{2}}|Z(x,w)|^{\frac{1}{2}}(\mathrm{i}X(x)^{-1}\,\mathrm{d}x + X(x)\,Z(x,w)^{-1}(\epsilon_2\,\mathrm{d}u + p(w)\,\mathrm{d}v),$$

where

$$Z(w,x) = \epsilon_1\,p(w) - \epsilon_2\,m(x), \quad g_0 = [\tfrac{1}{2}(1+f_0^2)]^{\frac{1}{2}}, \quad \epsilon_1 = \cos\gamma, \quad \epsilon_2 = \sin\gamma$$

and

$$p(w) = (c^2w^2+k^2+l^2)\cos\gamma + 2ckw, \quad m(x) = -(c^2x^2+k^2+l^2)\sin\gamma - 2clx,$$

$$W(w)^2 = -\tfrac{1}{3}\lambda c^2w^4\cos^2\gamma - \tfrac{4}{3}\lambda ckw^3\cos\gamma + f_2\,w^2 + f_1\,w + f_0,$$

$$X(x)^2 = -\tfrac{1}{3}\lambda c^2x^4\sin^2\gamma - \tfrac{4}{3}\lambda clx^3\sin\gamma - [f_2 + 2\lambda(k^2+l^2)]\,x^2 + g_1\,x + g_0,$$

where $\lambda, f_0, f_1, g_0, g_1, k, l, c$ and γ are constants. The metric is given by $g_{ij} = 2(l_{(i}n_{j)} - m_{(i}m^*{}_{j)})$ and is a solution of the Einstein field equations with cosmological constant. The Dirac equation admits a solution separable in the form

$$\psi = e^{i(\lambda^3 u + \lambda^4 v)} \begin{bmatrix} e^{i\beta} H_1(x) K_2(w) \\ e^{i\beta} H_2(x) K_1(w) \\ e^{-i\beta} H_1(x) K_1(w) \\ e^{-i\beta} H_2(x) K_2(w) \end{bmatrix},$$

where $e^{i\beta} = Z^{\frac{1}{4}}(h(w) - ig(x))^{-\frac{1}{2}}$, $h(w) = cw\cos\gamma + l$ and $g(x) = cx\sin\gamma + k$. The separated ordinary differential equations are given by

$$\tfrac{1}{\sqrt{2}}(W\partial/\partial w + f_0 g_0^{-2} W^{-1}(p\lambda^3 - \epsilon_2\lambda^4) + \tfrac{1}{2}W')K_1 - \lambda^1 h K_2 = \lambda^2 K_2,$$
$$\tfrac{1}{\sqrt{2}}(-f_0 W\partial/\partial w + g_0^{-2} W^{-1}(p\lambda^3 - \epsilon_2\lambda^4) - \tfrac{1}{2}f_0 W')K_2 - \lambda^1 h K_1 = -\lambda^2 K_1,$$
$$\tfrac{1}{\sqrt{2}}(-iX\partial/\partial x + X^{-1}(\epsilon_1\lambda^3 - m\lambda^4) - \tfrac{1}{2}iX')H_2 - i\lambda^1 g H_1 = -\lambda^2 H_1,$$
$$\tfrac{1}{\sqrt{2}}(iX\partial/\partial x + X^{-1}(\epsilon_1\lambda^3 - m\lambda^4) + \tfrac{1}{2}iX')H_1 - i\lambda^1 g H_2 = \lambda^2 H_2.$$

The notion of factorizable systems is not, however, sufficient for a comprehensive treatment of separable solutions of the Dirac equation. This is made clear by the example (Fels & Kamran 1990), of a space-time with the infinitesimal distance

$$ds^2 = dt^2 - a(t)^2(dx^2 + b(x)^2(dy^2 + c(y)^2\,dz^2)). \tag{6}$$

Metrics of this form include the various Robertson–Walker cosmological models. A suitable choice of null tetrad frame is

$$l_j\,dx^j = \tfrac{1}{\sqrt{2}}(dt + a(t)\,dx), \quad n_j\,dx^j = \tfrac{1}{\sqrt{2}}(dt - a(t)\,dx), \quad m_j\,dx^j = \tfrac{1}{\sqrt{2}}a(t)\,b(x)\,(dy + ic(y)\,dz).$$

The non-zero spin coefficients are

$$\epsilon = -\gamma = a'(t)/2\sqrt{2}a(t), \quad \rho = -(1/\sqrt{2}a(t))\,(a'(t) + b'(x)/b(x))$$
$$\mu = (1/\sqrt{2}a(t))\,(a'(t) - b'(x)/b(x)), \quad \alpha = -\beta = c'(y)/2\sqrt{2}a(t)\,b(x)\,c(y).$$

The metric given here has a Petrov type D Weyl tensor with l^i, n^i aligned with its repeated null directions. The non-zero Weyl scalar is given by

$$\Psi_2 = (c(y)\,(b'(x)^2 - b(x)\,b''(x)) + c''(y))/6a(t)^2 b(x)^2 c(y).$$

Separable solutions of Dirac's equation when written in spinor form relative to this null tetrad frame are

$$\varphi_0 = A_1(t)\,B_1(x)\,C_1(y)\,e^{i\lambda^4 z}, \quad \varphi_1 = A_1(t)\,B_2(x)\,C_2(y)\,e^{i\lambda^4 z},$$
$$\chi_{0'} = A_2(t)\,B_1(x)\,C_1(y)\,e^{i\lambda^4 z}, \quad \chi_{1'} = A_2(t)\,B_2(x)\,C_2(y)\,e^{i\lambda^4 z},$$

where the functions appearing in the above solutions satisfy the coupled set of ordinary differential equations

$$a(t)\,dA_2/dt - i\sqrt{2}\lambda^1 A_1 + \tfrac{3}{2}a'(t)\,A_2 = i\lambda^2 A_2,$$
$$a(t)\,dA_1/dt - i\sqrt{2}\lambda^1 A_2 - \tfrac{3}{2}a'(t)\,A_1 = -i\lambda^2 A_1,$$
$$b(x)\,dB_1/dx + b'(x)\,B_1 - ib(x)\,\lambda^2 B_1 = i\lambda^3 B_2,$$
$$-b(x)\,dB_2/dx - b'(x)\,B_1 - ib(x)\,\lambda^2 B_2 = i\bar{\lambda}^3 B_1,$$

$$\frac{dC_2}{dy} - \frac{\lambda^4}{c(y)}C_2 + \frac{c'(y)}{2c(y)}C_2 = -i\bar{\lambda}^3 C_1, \quad \frac{dC_1}{dy} + \frac{\lambda^4}{c(y)}C_1 + \frac{c'(y)}{2c(y)}C_1 = -i\lambda^3 C_2.$$

In contrast to the solution of the Dirac equation in the case of a Kerr–Newman space-time, it is not possible to characterize all the separation constants as the eigenvalues of first-order symmetry operators. In fact the space of first-order symmetry operators commuting with the Dirac operator is three dimensional in this case. This follows from the fact that for a general metric of the form (6) there is only one Killing–Yano tensor of each type: a valence one Killing–Yano tensor K^i with only one non-zero component $K^z = 1$, a valence two Killing–Yano tensor K^{ij} with only non-zero component $K^{yz} = 1/a(t)\,b(x)\,c(y)$ and a valence three Killing–Yano tensor K^{ijk} with only non-zero component $K^{xyz} = 1/a(t)^2 b(x)^2 c(y)$. The corresponding first-order symmetry operators can be read off from the general form given in (3):

$$K = K^i I_4 \nabla_i - \tfrac{1}{4}\nabla_l K_j \gamma^{jl}, \quad L = K^i_j \gamma^5 \gamma^j \nabla_i + \tfrac{1}{3}(\nabla^*_l K^l_m)\gamma^m, \quad N = K^i_{jk}\gamma^{jk}\nabla_i - \tfrac{3}{4}(\nabla^*_l K^l)\gamma^5.$$

The corresponding Dirac spinor solutions are eigensolutions of the operators K, N and L^2 with corresponding eigenvalues $\mathrm{i}\lambda^4$, $\mathrm{i}\lambda^2$ and $\lambda^3\bar{\lambda}^3$. These operators together with the Dirac operator $H = \gamma^l \nabla_l$ all commute. (Here a complete characterization of the separation parameters requires second-order symmetry operators. Examples of separation for the Dirac equation are known that require even higher-order symmetries.)

Recently Iyer & Kamran (1991) have studied separation of variables for the Dirac equation in an extended class of lorentzian metrics with rotational symmetry. Explicit solutions of the Dirac equation have been extensively studied in Minkowski space-time. For a description of this work see Bagrov & Gitman (1990) and Shiskin & Villalba (1989). In particular, Shiskin's approach to obtaining separable coordinates and eigenvalue characterizations holds promise for extension to curved space-times.

4. Spin $\frac{1}{2}$, 1 and $\frac{3}{2}$ equations in a curved space-time background

Equations that have explicit solution by separation techniques similar to those for the Dirac equation are the neutrino equation, Maxwell's equations and the Rarita–Schwinger spin $\frac{3}{2}$ equations. We consider in all cases the Kerr space-time background and, in the case of Maxwell's equations, Robertson–Walker type space-time also.

(a) *The neutrino equation*

The neutrino equation is in fact a special form of the Dirac equation for which $m_{\mathrm{e}} = 0$:

$$\nabla^{AA'}\varphi_A = 0.$$

The eigenvalue equation for the non-trivial symmetry operator is

$$L_B{}^{A'} N^A{}_{A'}\varphi_A = -\tfrac{1}{2}\lambda^2\varphi_B.$$

Kamran & McLenaghan (1984*b*) have shown that the Weyl neutrino equation is separable in the class of solutions which are Petrov type D electrovac with non-singular aligned Maxwell field satisfying the generalized Goldberg–Sachs theorem and that within this class of space-times the Dirac equation is separable in Carter's [A] family of solutions and in the A_0 null orbit solution of Debever & McLenaghan (1981).

(*b*) *Maxwell's equations*

These equations in spinor form are $\nabla^{A}_{A'}\varphi_{AB} = 0$, $\varphi_{AB} = \varphi_{BA}$. The separated components of two of the three distinct components of φ_{AB} have the form

$$\varphi_{00} = R_1 S_1 \mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)}, \quad \varphi_{11} = 2(\bar{\rho}^*)^2 R_{-1} S_{-1} \mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)}, \tag{7}$$

where $R_{\pm 1}$, $S_{\pm 1}$ are *Teukolsky functions*, i.e. they satisfy the second-order ordinary differential equations

$$(\Delta D_1 D_1^\dagger - 2\mathrm{i}\sigma r - \lambda) R_{+1} = 0, \qquad (\Delta D_0^\dagger D_0 + 2\mathrm{i}\sigma r + \lambda) R_{-1} = 0,$$
$$(L_0^\dagger L_1 + 2\sigma a \cos\theta + \lambda) S_{+1} = 0, \quad (L_0 L_1^\dagger - 2\sigma a \cos\theta - \lambda) S_{-1} = 0.$$

From these equations and by suitable normalization we obtain the so-called Teukolsky identities for spin 1:

$$\Delta D_0 D_0 R_{-1} = C \Delta R_{+1}, \quad \Delta D_0^\dagger D_0^\dagger \Delta R_{+1} = C^* R_{-1}, \quad L_0 L_1 S_{+1} = C S_{-1},$$
$$L_0^\dagger L_1^\dagger S_{-1} = C S_{+1}, \quad C^2 = \lambda^2 - 4\alpha^2\sigma^2, \quad \alpha^2 = a^2 + am/\sigma.$$

One can then obtain

$$\varphi_{01} = (1/\sqrt{2}\bar{\rho}^* C)\,[D_0 L_1 - (1/\bar{\rho}^*)\,(L_1 + \mathrm{i}a \sin\theta D_0)]\, R_{-1} S_{+1}.$$

The constants λ and C have been shown by Kalnins *et al.* (1989*a*) to be characterized by eigenvalue equations of symmetry operators

$$\Lambda^{C}_{(A}\phi_{B)\,C} = [K_{(A|}^{A'EE'}\nabla_{EE'} - M^{A'}_{(A|}]\,[K^{CDD'}_{A'}\nabla_{DD'} + 2M^{C}_{A'}]\,\phi_{|B)\,C} = \tfrac{1}{2}\lambda\phi_{AB},$$
$$C^{AB}_{A'B'}\phi_{AB} = [K^{AEE'}_{(A'|}\nabla_{EE'} + M^{A}_{(A'|}]\,[K^{B}_{|B')}{}^{DD'}\nabla_{DD'} + 2M^{B}_{|B')}]\,\phi_{AB} = C\bar{\phi}_{A'B'}.$$

As with the Dirac and Weyl neutrino equations, the Killing–Yano tensor and its associated Killing tensor play a crucial role in the separation of Maxwell's equations. These equations can also be solved for the Robertson–Walker type space-times. Indeed, solutions can be found of the form

$$\varphi_{00} = A(t)\,h_0(x)\,g_0(y)\,\mathrm{e}^{-\mathrm{i}\lambda z}, \quad \varphi_{01} = A(t)\,h_1(x)\,g_1(y)\,\mathrm{e}^{-\mathrm{i}\lambda z}, \quad \varphi_{11} = A(t)\,h_2(x)\,g_2(y)\,\mathrm{e}^{-\mathrm{i}\lambda z},$$

where the separation functions satisfy

$$\left[\frac{\partial}{\partial y} - \frac{\lambda}{c(y)} + \frac{c'(y)}{c(y)}\right] g_0(y) = \lambda_4 g_1(y), \quad \left[\frac{\partial}{\partial y} - \frac{\lambda}{c(y)}\right] g_1(y) = \lambda_3 g_2(y),$$

$$\left[\frac{\partial}{\partial y} + \frac{\lambda}{c(y)}\right] g_1(y) = \lambda_2 g_0(y), \quad \left[\frac{\partial}{\partial y} + \frac{\lambda}{c(y)} + \frac{c'(y)}{c(y)}\right] g_2(y) = \lambda_1 g_1(y),$$

$$\frac{\lambda_4}{b(x)} h_0 + \left[u + \frac{\partial}{\partial x} - 2\frac{b'(x)}{b(x)}\right] h_1 = 0, \quad \frac{\lambda_3}{b(x)} h_1 + \left[u + \frac{\partial}{\partial x} - \frac{b'(x)}{b(x)}\right] h_2 = 0,$$

$$\left[u - \frac{\partial}{\partial x} - \frac{b'(x)}{b(x)}\right] h_0 + \frac{\lambda_2}{b(x)} h_1 = 0, \quad \left[u - \frac{\partial}{\partial x} - 2\frac{b'(x)}{b(x)}\right] h_1 + \frac{\lambda_1}{b(x)} h_2 = 0.$$

These equations are consistent if $\lambda_1\lambda_3 = \lambda_2\lambda_4$. For a suitable choice of frame it is also possible to solve the massive spin 1 equations in Robertson–Walker type space-times. It is not, however, possible to proceed in this manner for free field equations of spin greater than 1. The reason for this is that the whole process developed for these space-times directly mimics the procedure used in Minkowski space-time, where group theory guarantees the results (Gel'fand *et al.* 1963). The recurrence formulas among the matrix elements that are necessary to achieve separation in that case can only be mimicked for general Robertson–Walker metrics for spins 1, $\frac{1}{2}$ or 0. Any attempt to mimic these methods of solution for higher-spin equations requires that the metric

$$d\bar{s}^2 = dx^2 + b^2(x)\,(dy^2 + c^2(y)\,dz^2)$$

correspond to a space of constant curvature. This occurs in the case of the various Robertson–Walker cosmological models as has been shown by Kalnins & Miller (1991).

(c) *The Rarita–Schwinger equation*

The separation and solution of the equations for various scalars associated with a Rarita–Schwinger field in a curved space-time was first achieved by Güven (1980). Güven, in proving the no-hair conjecture for the uncharged black holes of supergravity theory, showed that all solutions of the Rarita–Schwinger equation in the Kerr space-time that cannot be transformed away via a supersymmetry transformation are expressible in terms of Teukolsky functions for $s = \pm\frac{3}{2}$. Kamran (1985) later showed the separability of the Rarita–Schwinger equation for all type D vacuum space-times. The Rarita–Schwinger equation can be written in spinor form as $\nabla^A_{A'}\theta_{ABB'} = 0$, where $\theta_{ABB'} = \theta_{(AB)B'}$. One can construct a new field ϕ_{ABC} from the Rarita–Schwinger field by

$$\phi_{ABC} = \nabla_{(AA'}\theta^{A'}_{BC)}. \tag{8}$$

It is not too difficult to show that the new field ϕ_{ABC} must satisfy

$$\nabla^A_{A'}\phi_{ABC} = \Psi_{BCKL}\theta^{KL}_{A'} - \Phi_{K(BAK'}\theta^{KK'}_{C)} - \Lambda\theta_{BCA'}, \tag{9}$$

where Ψ, Φ and Λ, respectively, are the components of the Riemann curvature spinor (Penrose & Rindler 1984, 1986). At this point we will abandon the Rarita–Schwinger equation and consider the more general coupled system formed by equations (8) and (9). In a vacuum we have then

$$\nabla^A_{A'}\phi_{ABC} = \Psi^{KL}_{BC}\theta_{KLA'}, \quad \nabla_{(AA'}\theta^{A'}_{BC)} = \phi_{ABC}.$$

This system is consistent in a (vacuum) space-time. Note that any solution $\theta_{ABB'}$ of the Rarita–Schwinger equation will be a solution of this coupled system (although the converse is not true in general).

In a Petrov type D vacuum space-time and with a canonical tetrad, the above pair of equations give rise to the following three equations involving ϕ_0:

$$(D - \epsilon - 3\rho)\,\phi_1 - (\bar{\delta} + \pi - 3\alpha)\,\phi_0 = -\Psi_2\,\theta_{000'},$$
$$(\delta - \beta - 3\tau)\,\phi_1 - (\tilde{\Delta} - 3\gamma + \mu)\,\phi_0 = -\Psi_2\,\theta_{001'},$$
$$(D - \bar{\rho})\,\theta_{001'} - (\delta - a^* - 2\beta + \pi^*)\,\theta_{000'} = \phi_0.$$

In Kerr space-time and with

$$\Phi_0 = \phi_0, \quad \Phi_1 = \bar{\rho}^*\phi_1, \quad \eta_{000'} = \surd 2\bar{\rho}^*\theta_{000'}, \quad \eta_{001'} = \surd 2\bar{\rho}\bar{\rho}^*\theta_{001'},$$

these three equations become

$$[D_0 + 2/\bar{\rho}^*]\,\Phi_1 - [L_{\frac{3}{2}} - (2ia/\bar{\rho}^*)\sin\theta]\,\Phi_0 = -\Psi_2\,\eta_{000'},$$
$$[L^{\dagger}_{-\frac{1}{2}} + (2ia/\bar{\rho}^*)\sin\theta]\,\Phi_1 + \Delta[D^{\dagger}_{\frac{3}{2}} - 2/\bar{\rho}^*]\,\Phi_0 = -\Psi_2\,\eta_{001'},$$
$$[D_0 - 1/\bar{\rho}^*]\,\eta_{001'} - [L^{\dagger}_{-\frac{1}{2}} - (ia/\bar{\rho}^*)\sin\theta]\,\eta_{000'} = 2\bar{\rho}^*\bar{\rho}^*\Phi_0.$$

Together these equations imply that Φ_0 satisfies the second-order separable equation

$$[\Delta D_1 D^{\dagger}_{\frac{3}{2}} + L^{\dagger}_{-\frac{1}{2}} L_{\frac{3}{2}} - 4\mathrm{i}\sigma\bar{\rho}]\,\Phi_0 = 0,$$

which admits the solution $\Phi_0 = R_{+\frac{3}{2}} S_{+\frac{3}{2}}$. The functions $R_{+\frac{3}{2}}$ and $S_{+\frac{3}{2}}$ satisfy the Teukolsky equations

$$(\Delta D_1 D^{\dagger}_{\frac{3}{2}} - 4\mathrm{i}\sigma r)\,R_{+\frac{3}{2}} = \lambda R_{+\frac{3}{2}}, \quad (L^{\dagger}_{-\frac{1}{2}} L_{\frac{3}{2}} + 4\sigma a\cos\theta)\,S_{+\frac{3}{2}} = -\lambda S_{+\frac{3}{2}}.$$

A solution can now be found for the rest of the components of the two fields. Such a solution is

$$\Phi_0 = R_{+\frac{3}{2}} S_{+\frac{3}{2}},$$

$$\eta_{000'} = (1/\Psi_2)\,[L_{\frac{3}{2}} - (2\mathrm{i}a/\bar{\rho}^*)\sin\theta]\,R_{+\frac{3}{2}} S_{+\frac{3}{2}}, \quad \eta_{001'} = -(1/\Psi_2)\,\Delta[D^{\dagger}_{\frac{3}{2}} - 2/\bar{\rho}^*]\,R_{+\frac{3}{2}} S_{+\frac{3}{2}}$$

with the remaining components of the two fields being zero. By considering the equations in which ϕ_3 appears, one can construct a second solution

$$\Phi_3 = R_{-\frac{3}{2}} S_{-\frac{3}{2}},$$

$$\eta_{110'} = -(1/\Psi_2)\,[D_0 - 2/\bar{\rho}^*]\,R_{-\frac{3}{2}} S_{-\frac{3}{2}}, \quad \eta_{111'} = -(1/\Psi_2)\,[L^{\dagger}_{\frac{3}{2}} - (2\mathrm{i}a/\bar{\rho}^*)\sin\theta]\,R_{-\frac{3}{2}} S_{-\frac{3}{2}},$$

$$\Phi_3 = \bar{\rho}^{*3}\phi_3, \quad \Phi_2 = \bar{\rho}^{*2}\phi_2, \quad \eta_{110'} = \sqrt{2}\bar{\rho}^{*3}\theta_{110'}, \quad \eta_{111'} = \sqrt{2}\bar{\rho}\bar{\rho}^{*3}\theta_{111'},$$

where the Teukolsky functions $R_{-\frac{3}{2}}$ and $S_{-\frac{3}{2}}$ satisfy the equations

$$(\Delta D^{\dagger}_{-\frac{1}{2}} D_0 + 4\mathrm{i}\sigma r)\,R_{-\frac{3}{2}} = \lambda R_{-\frac{3}{2}}, \quad (L_{-\frac{1}{2}} L^{\dagger}_{\frac{3}{2}} - 4\sigma a\cos\theta)\,S_{-\frac{3}{2}} = -\lambda S_{-\frac{3}{2}}.$$

5. Generalized Hertz potentials

The use of generalized Hertz potentials to produce solutions to various equations in algebraically special space-times is due to Cohen & Kegeles (1979). A generalized Hertz potential is a totally symmetric $2s$-spinor $\bar{P}^{A'_1 A'_2 \ldots A'_{2s}}$ satisfying

$$\nabla_{AX'}\nabla^{AX'}\bar{P}^{A'_1 A'_2 \ldots A'_{2s}} - 2\nabla^{B(A'_1}G_B{}^{A'_2\ldots A'_{2s})} - (2s-1)\,s\bar{\Psi}_{B'C'}{}^{(A'_1A'_2}\bar{P}^{A'_3\ldots A'_{2s})B'C'} = 0,$$

where $G_B{}^{A'_2\ldots A'_{2s}}$ is an arbitrary gauge field and Ψ is the Weyl spinor. One can construct a spin s field from the potential and gauge field as follows:

$$\phi_{A_1A_2\ldots A_{2s}} = \nabla_{(A_1A'_1}\nabla_{A_2A'_2}\cdots\nabla_{A_{2s-1}A'_{2s-1}}[\nabla_{A_{2s}A'_{2s}}\bar{P}^{A'_1A'_2\ldots A'_{2s}} - G_{A_{2s})}{}^{A'_1A'_2\ldots A'_{2s-1}}].$$

In a flat space-time the resulting field $\phi_{A_1\ldots A_{2s}}$ will satisfy the massless spin s field equation $\nabla^B{}_{B'}\phi_{BA_2\ldots A_{2s}} = 0$. However, when the space-time is curved the field $\phi_{A_1A_2\ldots A_{2s}}$ satisfies the massless spin s field equations for $s = \frac{1}{2}$ and $s = 1$ only. For $s = 2$ the method of generalized Hertz potentials can be used to construct a solution for the linear gravitational perturbations of the Kerr space-time. Solving the equation for the generalized Hertz potential via separation of variables depends crucially on making the right choice of gauge field.

For the Weyl neutrino equation the corresponding Hertz potential $\bar{P}^{A'}$ satisfies the equation $\nabla^{BA'}\nabla_{BB'}\bar{P}^{B'} - \nabla^{BA'}G_B = 0$, where G_B is an arbitrary gauge field. The new field is constructed as follows: $\varphi_A = \nabla_{AA'}\bar{P}^{A'} - G_A$. In the Kerr black hole space-time a convenient choice of gauge field is $G_B = \bar{U}_{BB'}\bar{P}^{B'}$ where $\bar{U}_{BB'}$ is the vector having the components

$$\bar{U}_{00'} = \rho^*, \quad \bar{U}_{01'} = -\pi^*, \quad \bar{U}_{10'} = \tau^*, \quad \bar{U}_{11'} = -\mu^*.$$

This choice leads to Hertz potentials with solutions

$$\bar{P}^{0'} = R_{-\frac{1}{2}} S_{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)}, \quad \bar{P}^{1'} = 0 \quad \text{or} \quad \bar{P}^{1'} = \bar{\rho} R_{\frac{1}{2}} S_{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}(m\varphi+\sigma t)}, \quad \bar{P}^{0'} = 0,$$

where $R_{\pm\frac{1}{2}}$, $S_{\pm\frac{1}{2}}$ are Teukolsky functions obtained from the separation equations for Dirac's equation by setting $m_e = 0$. Both possible choices of Hertz potential give rise to the same neutrino field.

For the spinor form of Maxwell's equations a similar potential can be found by writing

$$\varphi_{AB} = \nabla_{(AA'}[\nabla_{B)B'}\bar{P}^{A'B'} - G_{B)}{}^{A'}].$$

The new field satisfies Maxwell's equations in spinor form. With the specific choice of gauge field $G_C{}^{B'} = -2\bar{U}_{CC'}\bar{P}^{C'B'}$ (Kerr black hole) the form of the Maxwell field given in (7) is recovered with the Hertz potentials being of the form

$$\bar{P}^{0'0'} = R_{-1}S_{+1}\,e^{i(\sigma t+m\varphi)}, \qquad \bar{P}^{0'1'} = \bar{P}^{1'1'} = 0,$$

or

$$\bar{P}^{1'1'} = \bar{\rho}^2 R_{+1}S_{-1}\,e^{i(\sigma t+m\varphi)}, \quad \bar{P}^{0'1'} = \bar{P}^{0'0'} = 0.$$

Both these forms of the Hertz potential lead to the same Maxwell spinors. An interesting feature of the Hertz potential equation is that the choice of gauge function $G_B{}^{A'_2\ldots A'_{2s}} = -2sU_{BB'}\bar{P}^{B'A'_2\ldots A'_{2s}}$ yields solutions whose only non-zero components are of the form

$$\bar{P}^{0'\ldots 0'} = R_{-s}S_s\,e^{i(m\varphi+\sigma t)}, \quad \text{or} \quad \bar{P}^{1'\ldots 1'} = \bar{\rho}^{2s}R_sS_{-s}\,e^{i(m\varphi+\sigma t)},$$

where $R_{\pm s}$, $S_{\pm s}$ are the spin s Teukolsky functions satisfying the second-order equations

$$\begin{aligned}(\Delta D^{\dagger}_{1-s}D_0 + 2(2s-1)\,i\sigma r)R_{-s} = \lambda R_{-s}, \quad (\Delta D_1 D^{\dagger}_s - 2(2s-1)\,i\sigma r)R_s = \lambda R_s,\\ (L^{\dagger}_{1-s}L_s + 2(2s-1)\,\sigma a\cos\theta)S_{+s} = -\lambda S_{+s}, \quad (L_{1-s}L^{\dagger}_s - 2(2s-1)\,\sigma a\cos\theta)S_{-s} = -\lambda S_{-s}.\end{aligned} \tag{10}$$

These potential functions can be used to obtain components of the perturbed metric tensor corresponding to gravitational perturbations. The perturbed Einstein vacuum field equations $\delta R_{\alpha\beta} = 0$ take the form

$$\nabla_\rho\nabla^\rho h_{\alpha\beta} - \nabla_\rho\nabla_{(\beta}h^\rho_{\alpha)} + \nabla_\alpha\nabla_\beta h = 0, \quad h = h^\mu_\mu, \quad h_{\mu\nu} = h_{\nu\mu}. \tag{11}$$

Here $h_{\nu\mu}$ is the perturbed metric tensor. This can be thought of also as a spin 2 wave equation. In terms of the Hertz potential and gauge spinors the perturbed metric is given by

$$\begin{aligned}h^{\pm}_{CDM'N'} = (\nabla_{(CP'}\nabla_{D)Q'}\bar{P}_{M'N'}{}^{P'Q'} - \nabla_{(CP'}G_{D)M'N'}{}^{P'})\\ \pm(\nabla_{E(M'}\nabla_{FN')}\bar{P}^{EF}{}_{CD} - \nabla_{E(M'}G_{N')CD}{}^{E})\end{aligned}$$

and the perturbed Weyl tensor is given by $\Psi_{ABCD} = \nabla_{(AW'}\nabla_{BX'}h^{W'X'}{}_{CD)}$. The explicit expressions for the perturbed metric tensor are

$$\begin{aligned}h^{\pm}_{\mu\nu} = \{&-l_\mu l_\nu[(\delta^* + \alpha + 3\beta^* - \tau^*)(\delta^* + 4\beta^* + 3\tau^*) + m^*_\mu m^*_\nu(D - \rho^*)(D + 3\rho^*)] + l_{(\mu}m^*{}_{\nu)}\\ &[(D + \rho - \rho^*)(\delta^* + 4\beta^* + 3\tau^*) + (\delta^* - \alpha + 3\beta^* - \pi - \tau^*)(D + 3\rho^*)]\}R_{-2}S_2\,e^{i(\sigma t+m\varphi)}\\ &\pm\{-l_\mu l_\nu[(\delta + \alpha^* + 3\beta - \tau)(\delta + 4\beta + 3\tau) + m_\mu m_\nu(D - \rho)(D + 3\rho)] + l_{(\mu}m_{\nu)}\\ &[(D + \rho^* - \rho)(\delta + 4\beta + 3\tau) + (\delta - \alpha^* + 3\beta - \pi^* - \tau)(D + 3\rho)]\}R_{-2}S_{-2}\,e^{i(\sigma t+m\varphi)},\end{aligned}$$

where the choice of gauge is $l^\mu h_{\mu\nu} = 0$, $h = 0$. An alternative method of solution of the gravitational perturbation problem has been given in Chandrasekhar's book (1983). In his solution the expressions for the perturbed components of the metric are more

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complex. A question that remains unanswered is for what operator can the perturbed components of the metric be represented as eigenfunctions with eigenvalue λ? In addition to the two solutions given above, is it possible to construct solutions of the gravitational perturbation equations in the de Donder gauge: $\nabla^{\mu} h_{\mu\nu} = 0$, $h = 0$? In his book Chandrasekhar investigated the problem of the combined electromagnetic and gravitational perturbations for the case of a Kerr–Newman black hole. He was unable to achieve a decoupling analogous to that achieved for the Kerr black hole. One way to approach this problem would be to look for a second-order symmetry for the combined system of perturbation equations. If such exists then it would indicate the possibility of obtaining closed form solutions for this problem. This matter is currently being investigated.

We have seen that the solution of systems of differential equations that arise in general relativity via some sort of separation of variables contains elegant examples but as yet no theoretical basis. More work needs to be done to improve this situation as well as to solve some of the problems specific to general relativity itself, e.g. the combined perturbation problem for the Kerr–Newman black hole.

Appendix A. Teukolsky functions and Teukolsky–Starobinsky identities

The Teukolsky functions $R_{\pm s}$ and $S_{\pm s}$, which satisfy the second-order differential equations (10), have a number of interesting properties. The equations satisfied by the $S_{\pm s}$ functions are forms of the confluent Heun equation. Expansion theorems and studies of the spectrum of the parameter λ and its various asymptotic forms have been worked out by Fackerell & Crossman (1977) and Leaver (1986). In particular it is possible to generalize the properties given above for the case of the Maxwell field and, implicitly, for the spin $\frac{1}{2}$ field with $m_{\mathrm{e}} = 0$. The relations which generalize these for spin s are the so-called Teukolsky–Starobinsky identities. We first note that R_{-s} and $\Delta^s R_{+s}$ satisfy complex conjugate equations. Similarly $S_{+s}(\pi-\theta)$ satisfies the same equation as $S_{-s}(\theta)$.

The following two identities (Kalnins *et al.* 1989*b*) hold for $s = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$:

$$\Delta^s D_0^{2s}(\Delta D_{1-s}^{\dagger} D_0 + 2(2s-1)\,\mathrm{i}\sigma r) = (\Delta D_{1-s} D_0^{\dagger} - 2(2s-1)\,\mathrm{i}\sigma r)\,\Delta^s D_0^{2s}, \tag{12}$$

$$L_{1-s} L_{2-s} \ldots L_{s-1} L_s (L_{1-s}^{\dagger} L_s + 2(2s-1)\,\sigma a \cos\theta) = (L_{1-s} L_s^{\dagger} - 2(2s-1)\,\sigma a\cos\theta)\, L_{1-s} L_{2-s} \ldots L_{s-1} L_s.$$

Two very similar identities are obtained by taking the complex conjugate of the first identity ($D \leftrightarrow D^{\dagger}$), and by letting $\theta \to \pi - \theta$ in the second identity ($L \leftrightarrow -L^{\dagger}$):

$$\Delta^s D_0^{\dagger 2s}(\Delta D_{1-s} D_0^{\dagger} - 2(2s-1)\,\mathrm{i}\sigma r) = (\Delta D_{1-s}^{\dagger} D_0 + 2(2s-1)\,\mathrm{i}\sigma r)\,\Delta^s D_0^{\dagger 2s}, \tag{13}$$

$$L_{1-s}^{\dagger} L_{2-s}^{\dagger} \ldots L_{s-1}^{\dagger} L_s^{\dagger} (L_{1-s} L_s^{\dagger} - 2(2s-1)\,\sigma a\cos\theta) = (L_{1-s}^{\dagger} L_s + 2(2s-1)\,\sigma a \cos\theta)\, L_{1-s}^{\dagger} L_{2-s}^{\dagger} \ldots L_{s-1}^{\dagger} L_s^{\dagger}.$$

A direct consequence of these is the following. If one acts on the function R_{-s} with both sides of (12) then one finds that $\Delta^s D_0^{2s} R_{-s}$ is a solution of the Teukolsky equation for $\Delta^s R_{+s}$. Similarly by acting on the function $\Delta^s R_{+s}$ with both sides of (13) one finds that $\Delta^s D_0^{\dagger 2s} \Delta^s R_{+s}$ is a solution of the Teukolsky equation for R_{-s}. By suitably choosing the relative normalization of the functions R_{-s} and R_{+s} we can write

$$\Delta^s D_0^{2s} R_{-s} = C_s\,\Delta^s R_{+s}, \quad \Delta^s D_0^{\dagger 2s} \Delta^s R_{+s} = C_s^{*} R_{-s},$$

where C is a complex constant.

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A similar treatment of the angular identities results in the relations

$$L_{1-s}L_{2-s}\dots L_{s-1}L_sS_{+s}=B_sS_{-s},\quad L^{\dagger}_{1-s}L^{\dagger}_{2-s}\dots L^{\dagger}_{s-1}L^{\dagger}_sS_{-s}=\bar{B}_sS_{+s},$$

where B_s is a real constant ($\bar{B}_s=(-1)^sB_s$). Here C_s and B_s are referred to as Starobinsky constants.

By use of the mapping $r\leftrightarrow\omega=\mathrm{i}a\cos\theta$ one can make the above radial and angular identities almost formally identical. By such means one can show that $B_s^2=(-1)^{2s}|C_s|^2|_{M=0}$. Computation of $|C_s|^2$ can be achieved by examining the identity $\Delta^sD_0^{\dagger 2s}\Delta^sD_0^{2s}R_{-s}=|C_s|^2$ and replacing the occurrences of second-order (and higher) derivatives of R_{-s} with first and zero-order terms by means of the Teukolsky equation that R_{-s} must satisfy. Such a procedure is very protracted for all but the first few small values of s. Chandrasekhar (1991) has shown how to determine the general expression for the Teukolsky–Starobinsky identity $|C_s|^2$ in the form of a determinant resulting from considerations following from the notion of algebraically special solutions. The Starobinsky constants for spin $\frac{1}{2}$ and 1 have already been mentioned. For spin $\frac{3}{2}$ and 2 they are

$$|C_{\frac{3}{2}}|^2=\lambda^2(\lambda+1)-16\sigma^2(\lambda\alpha^2-a^2),$$
$$|C_2|^2=\lambda^2(\lambda+2)^2-8\sigma^2\alpha^2\lambda(5\lambda+6)+96\sigma^2a^2\lambda+144\sigma^4\alpha^4+144\sigma^2M^2,$$

where $\alpha^2=a^2+ma/\sigma$.

We note here that it is possible to invent a relativistically invariant equation system which has the general spin s Teukolsky functions appearing in their solution, in fact the coupled system of equations

$$\nabla^{AA'}\varphi_{AA_2\dots A_{2s}}=(2s-1)\,(s-1)\,\Psi^{BC}_{(A_2A_3}\theta^{A'}_{A_4\dots A_{2s})\,BC},$$
$$I\nabla_{(A_1A'}\theta^{A'}_{A_2\dots A_{2s})}-\tfrac{1}{6}(2s-3)\,\theta^{A'}_{(A_2\dots A_{2s}}\nabla_{A_1)\,A'}I=I\varphi_{A_1A_2\dots A_{2s}},$$

where $I=\Psi_{ABCD}\,\Psi^{ABCD}$. Typically this system has solutions of the form

$$\varphi_{0\dots 0}=R_sS_s\,\mathrm{e}^{\mathrm{i}(\sigma t+m\varphi)},\quad \theta^{0'}_{0\dots 0}=\frac{-1}{\surd 2\rho^2(2s-1)\,(s-1)\,\Psi_2}\Delta(D^{\dagger}_s-(2s-1)/\rho^*)\,\varphi_{0\dots 0},$$

$$\theta^{1'}_{0\dots 0}=\frac{-1}{\surd 2\rho^*(2s-1)\,(s-1)\,\Psi_2}(L^{\dagger}_s-(2s-1)\,(\mathrm{i}a/\rho^*)\sin\theta)\,\varphi_{0\dots 0}$$

with similar expressions existing for solutions with indices consisting of all 1s.

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